

Adiabatic invariants of the extended KdV equation

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Abstract

When the Euler equations for shallow water are taken to the next order, beyond KdV, momentum and energy are no longer exact invariants. (The only one is mass.) However, adiabatic invariants (AI) can be found. When the KdV expansion parameters are zero, exact invariants are recovered. Existence of adiabatic invariants results from general theory of near-identity transformations (NIT) which allow us to transform higher order nonintegrable equations to asymptotically equivalent (when small parameters tend to zero) integrable form. Here we present the direct method of calculations of adiabatic invariants. It does not need a transformation to a moving reference frame nor performing a near-identity transformation. Numerical tests show that deviations of AI from almost constant values are indeed small.

Keywords: Shallow water waves, nonlinear equations, invariants of KdV2 equation, adiabatic invariants

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1. Introduction

The Eulerian shallow water and long wavelength equations are considered difficult, since they involve conditions on an unknown surface. A simple equation that gets around this is due to Korteweg and de Vries [1, 2]. A small parameter is introduced. There are other models, see [2, 3, 4]. It is a well known fact that the Korteweg – de Vries equation (KdV) possesses an infinite number of invariants, see, e.g. [5, 6], *also known as integrals of motion* [7, 8]. The lowest order invariant assures volume conservation of the fluid (as the fluid is assumed to be incompressible this is equivalent to mass conservation). The second invariant is related to the conservation of the fluid momentum. The third order KdV invariant is related to the energy. This relation is not so obvious. As shown in [9, 10] energy of the fluid has an invariant form only in a particular reference frame moving with the velocity of sound. In the fixed frame energy has noninvariant form. However, it varies only by a small fraction, particularly when collisions of solitons occur, see [10].

Several authors have extended KdV to the second order (KdV2), e.g. [10, 11, 12, 13, 14, 15, 16]. *Here the term second order means the order of perturbation expansion with respect to small parameters.* However, this improved form is short of exactly conserved entities other than the ubiquitous mass law.

Several papers [7, 17, 18, 19, 20, 21, 22, 23, 24, 25] claim existence of higher invariants and integrability of second order KdV type equations. *In particular Benjamin and Olver [7] discussed Hamiltonian structure, symmetries and conservation laws for water waves. A near-identity transformation, introduced by Kodama [17] and then used by many authors, allows us to transform second order KdV to asymptotically equivalent Hamiltonian form. Here term the asymptotic equivalence means neglect of terms of higher order. In general the existence of Hamiltonian form for transformed equation supplies the full hierarchy of invariants, which appear to be adiabatic invariants for the original equation.*

If there are no exact invariants in the system one looks for adiabatic (approximate) ones, like in [26]. *Here we introduced a straightforward method to calculate adiabatic invariants, which allows us to find them directly from the original 'physical' equation (it also works for equations written in dimensional variables). Our method consists of the following: one proceeds with the second order equation (1) as with construction of KdV invariants and then uses addition of KdV, multiplied by small parameter, to cancel nonintegrable terms.*

In [27] it is shown that KdV2 equation for nonflat bottom [14, 15] has just one local conservation law which corresponds to the exact invariant of mass.

2. Adiabatic invariants

In the derivation of the KdV equation in the shallow water problem several assumptions are made. The fluid is assumed to be ideal and viscosity is neglected. Therefore if initial motion is irrotational it stays irrotational. Approximation leading to KdV and higher order nonlinear wave equations is correct when two parameters relating the wave amplitude a , average wavelength l and the water depth h are small and of the same order of magnitude. The geometry of the problem and definitions of two small parameters α and β are given in Fig. 1.

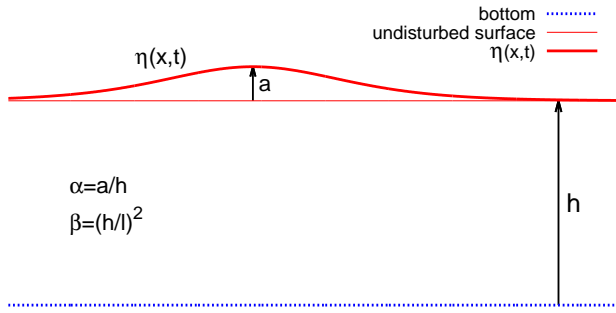


Figure 1: Schematic view of the geometry of the problem.

We consider the second order KdV equation [10, Eq. (1)], (α and β are small expansion parameters)

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} = 0, \quad (1)$$

called by Marchant and Smyth [11, Eq. (2.8)] the "extended KdV". The equation was considered by several authors, see, e.g. [11, 12, 13, 14, 15]. We will call it KdV2.

It is well known, see, e.g. [6, Ch. 5], that an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \quad (2)$$

where neither T (an analog to density) nor X (an analog to flux) contain partial derivatives with respect to t , corresponds to some *conservation law*. It can be applied, in particular, to the KdV equations and to the equations of KdV type like (1). Functions T and X may depend on $x, t, \eta, \eta_x, \eta_{2x}, \dots$, but not η_t . If both functions T and X_x are integrable on $(-\infty, \infty)$ and $\lim_{x \rightarrow \pm\infty} X = \text{const}$ (soliton solutions), then integration of equation (2) yields

$$\frac{d}{dt} \left(\int_{-\infty}^{\infty} T dx \right) = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} T dx = \text{const.} \quad (3)$$

$$\text{since} \quad \int_{-\infty}^{\infty} X_x dx = X(\infty, t) - X(-\infty, t) = 0. \quad (4)$$

The same conclusion applies for periodic solutions (cnoidal waves), when in the integrals (3), (4) limits of integration $(-\infty, \infty)$ are replaced by (a, b) , where $b - a = \Lambda$ is the space period of the cnoidal wave (the wave length).

In [10], we noted that $I^{(1)} = \int_{-\infty}^{\infty} \eta dx$ is an invariant of equation (1) and represents the conservation of mass like just as KdV.

2.1. Second invariant

The second invariant of KdV, $I^{(2)} = \int_{-\infty}^{\infty} \eta^2 dx$ is *not* an invariant of KdV2, since, see [10, Sec. III B], upon multiplication of equation (1) by η one obtains

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\frac{1}{2} \eta^2 \right) + \frac{\partial}{\partial x} \left[\frac{1}{2} \eta^2 + \frac{1}{2} \alpha \eta^3 + \frac{1}{6} \beta \left(-\frac{1}{2} \eta_x^2 + \eta \eta_{2x} \right) \right] \\ &\quad - \frac{3}{32} \alpha^2 \eta^4 + \frac{19}{360} \beta^2 \left(\frac{1}{2} \eta_{xx}^2 - \eta_x \eta_{3x} + \eta \eta_{4x} \right) + \frac{5}{12} \alpha \beta \eta^2 \eta_{2x} \\ &\quad + \frac{1}{8} \alpha \beta \eta \eta_x \eta_{2x}. \end{aligned} \quad (5)$$

The last term in (5) cannot be expressed as $\frac{\partial}{\partial x} X(\eta, \eta_x, \dots)$.

Therefore $\int_{-\infty}^{\infty} \eta^2 dx$ is not a conserved quantity. There are no exact higher order invariants of (1).

It is possible, however, to find approximate invariants of (1), for which terms violating the invariance are of the third order in α, β . Our method allows us to find such approximate invariants with relatively low effort. It consists in forming equation containing functions T and X by some manipulations with KdV2. Then some terms in X have no integrable form with respect to x like the last term in (5). We add some linear combination of type $(c_1\alpha + c_2\beta)KdV$ to that equation and require that nonintegrable terms cancel. This action yields a new T' function and an approximate conservation law for $\int_{-\infty}^{\infty} T' dx$.

The first approximate invariant can be obtained by adding to (5) equation (1) multiplied by $c_1\beta\eta_{2x}$, dropping third-order terms and choosing an appropriate value of c_1 in order to cancel the term $\frac{1}{8}\alpha\beta\eta\eta_x\eta_{2x}$. When this is done the expression

$$c_1\beta\eta_t\eta_{2x} + c_1\beta\eta_x\eta_{2x} + c_1\frac{3}{2}\alpha\beta\eta\eta_x\eta_{2x} + c_1\frac{1}{6}\beta^2\eta_{2x}\eta_{3x} \quad (6)$$

is left. In (6), the second and fourth terms are integrable with respect to x then can be omitted. The condition $c_1\frac{3}{2}\alpha\beta\eta\eta_x\eta_{2x} + \frac{1}{8}\alpha\beta\eta\eta_x\eta_{2x} = 0$ implies $c_1 = -\frac{1}{12}$.

Integration of the first term in (6) over x gives

$$\begin{aligned} \int_{-\infty}^{\infty} c_1\beta\eta_t\eta_{2x} dx &= c_1\beta \left(\eta_t\eta_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \eta_{tx}\eta_x dx \right) \\ &= -c_1\beta \int_{-\infty}^{\infty} \left(\frac{1}{2} \eta_x^2 \right) dx. \end{aligned} \quad (7)$$

Since terms with $\eta_x\eta_{2x}$ and $\eta_{2x}\eta_{3x}$ can be expressed as $\frac{\partial}{\partial x} \left(-\frac{1}{2} \eta_x^2 \right)$ and $\frac{\partial}{\partial x} \left(-\frac{1}{2} \eta_{2x}^2 \right)$, respectively, the final result is

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{1}{2} \left(\eta^2 + \frac{1}{12} \beta \eta_x^2 \right) dx + F(\eta, \eta_x, \eta_{2x}) \Big|_{-\infty}^{\infty} = O(\alpha^3), \quad (8)$$

where $F(\eta, \eta_x, \eta_{2x})$ comes from the flux term and vanishes due to properties of the solutions at $\pm\infty$. We assume that solutions at $\pm\infty$ vanish or else are periodic.

Therefore we have an approximate (adiabatic) invariant of KdV2 (1) in the form

$$I_{\text{ad}}^{(2\beta)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{12} \beta \eta_x^2 \right) dx \approx \text{const.} \quad (9)$$

There is an alternative way to cancel the last term in (5) and obtain a second approximate invariant. This approximate invariant can be calculated in the same way as above but with the factor $c_2 \alpha \eta^2$ replacing $c_1 \beta \eta_{2x}$. Then the relation corresponding to (6) is

$$c_2 \alpha \eta_t \eta^2 + c_2 \alpha \eta^2 \eta_x + c_2 \frac{3}{2} \alpha^2 \eta^3 \eta_x + c_2 \frac{1}{6} \alpha \beta \eta^2 \eta_{3x}. \quad (10)$$

In integration over x the last term in (10) can be transformed to $-\frac{1}{3} c_2 \alpha \beta \eta \eta_x \eta_{2x}$. The condition for cancelation of this term with $\frac{1}{8} \alpha \beta \eta \eta_x \eta_{2x}$ gives $c_2 = \frac{3}{8}$. Then the first term in (10) yields

$$c_2 \alpha \eta_t \eta^2 = \frac{\partial}{\partial t} \left(\frac{1}{8} \alpha \eta^3 \right) \quad (11)$$

and since the other terms are integrable we obtain an alternative approximate invariant of KdV2 ($\frac{1}{2}$ is omitted)

$$I_{\text{ad}}^{(2\alpha)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{4} \alpha \eta^3 \right) dx \approx \text{const.} \quad (12)$$

The existence of two independent adiabatic invariants $I_{\text{ad}}^{(2\alpha)}$ and $I_{\text{ad}}^{(2\beta)}$ means also that

$$I_{\text{ad}}^{(2)} = \epsilon I_{\text{ad}}^{(2\alpha)} + (1 - \epsilon) I_{\text{ad}}^{(2\beta)} \quad (13)$$

is an adiabatic invariant for any $\epsilon \in [0, 1]$.

2.2. Third invariant

In the KdV hierarchy the third invariant $I_3 = \int_{-\infty}^{\infty} \left(\eta^3 - \frac{1}{3} \eta_x^2 \right) dx = \text{const.}$ is usually referred to energy conservation [5, 6]. What is this constant? This is not clear even in the basic KdV equation. There it is a component of the mechanical energy, not the whole energy. Not even that in some coordinate systems [10].

It is possible to obtain next approximate invariant to KdV2 (1). For $\beta = \alpha$ it takes the following form

$$I_{\text{ad}}^{(3)} = \int_{-\infty}^{\infty} \left(\eta^3 - \frac{1}{3} \eta_x^2 \right) dx - \int_{-\infty}^{\infty} \alpha \eta^4 dx + \int_{-\infty}^{\infty} \frac{7}{12} \alpha \eta \eta_x^2 dx. \quad (14)$$

Note that the first term in (14) is identical to the exact KdV invariant.

The presented method allows, in principle, to obtain higher order adiabatic invariants.

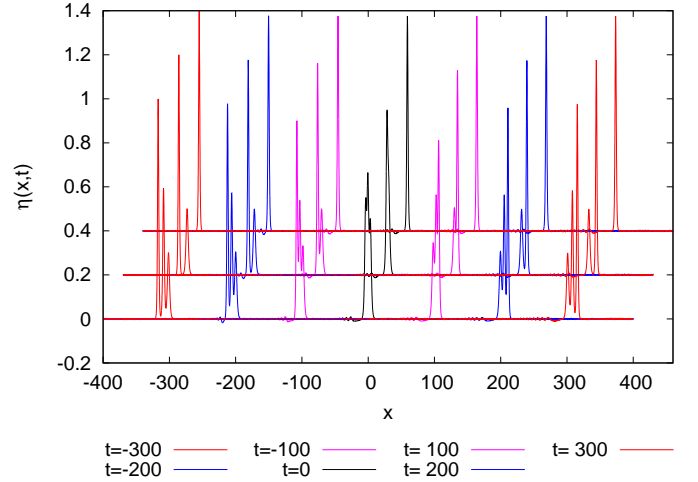


Figure 2: Time evolution of initially 1-, 2- and 3-soliton KdV solution according to KdV2 (1).

3. Numerical tests

One might wonder how good these invariants are. The calculations presented below give some insight. First we calculated the time evolution, governed by equation (1), for three particular waves. As initial conditions 1-, 2- and 3-soliton solutions of KdV were taken. The amplitudes of the 3-soliton solution were chosen to be 1.0, 0.6 and 0.3, the amplitudes of the 2-soliton solution were chosen as 1.0 and 0.3 and the amplitude of this single soliton was chosen as 1.0. The motion of these waves according to (1) and their shapes at some instants are presented in Fig. 2. In order to avoid overlaps, vertical shifts by 0.2 and horizontal shifts by 30 were applied in the figure. In all calculations presented here the small parameters were both $\alpha = \beta = 0.1$.

To study approximate invariants $I_{\text{ad}}^{(2\beta)}$ and $I_{\text{ad}}^{(2\alpha)}$ we write each of them as the sum of two terms.

$$I_{\text{ad}}^{(2\alpha)} = \int_{-\infty}^{\infty} \eta^2 dx + \int_{-\infty}^{\infty} \frac{1}{4} \alpha \eta^3 dx =: Ie(t) + Ia(t), \quad (15)$$

$$I_{\text{ad}}^{(2\beta)} = \int_{-\infty}^{\infty} \eta^2 dx + \int_{-\infty}^{\infty} \frac{1}{12} \beta \eta_x^2 dx =: Ie(t) + Ib(t). \quad (16)$$

The first term in (15) and (16) is identical to the exact KdV invariant.

Values of adiabatic invariants $I_{\text{ad}}^{(2\alpha)}$ (15) and $I_{\text{ad}}^{(2\beta)}$ (16) calculated for the time evolution of waves displayed in Fig. 2 are presented in Fig. 3. In this scale both adiabatic invariants look perfectly constant. In order to see how good these invariants are we show how they change with respect to the initial values.

Fig. 4 shows changes in the quantities Ie , Ia and Ib for all three 1-, 2-, and 3-soliton waves presented in Fig. 2. Displayed are the relative changes of

$$Ie = \frac{Ie(t) - Ie(0)}{Ie(0) + Ia(0)}, \quad Ia = \frac{Ia(t) - Ia(0)}{Ie(0) + Ia(0)}, \quad Ib = \frac{Ib(t) - Ib(0)}{Ie(0) + Ia(0)}.$$

The figure shows that the corrections Ia , Ib to the KdV invariant Ie have very similar absolute values as Ie but opposite sign. Therefore one can expect that their sums with Ie

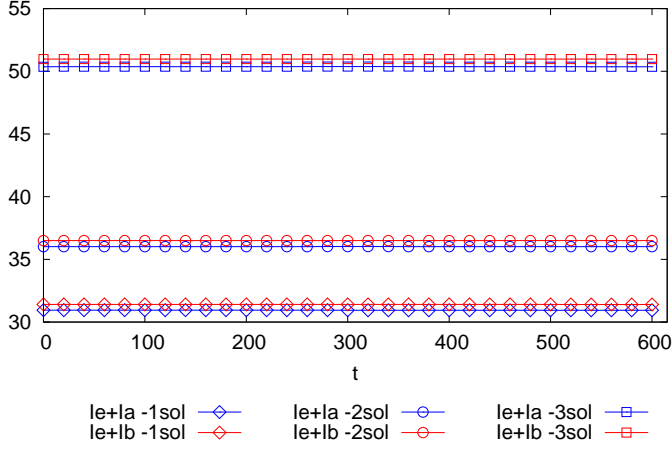


Figure 3: Absolute values of the adiabatic invariants (15) and (16) for the time evolution shown in Fig. 2.

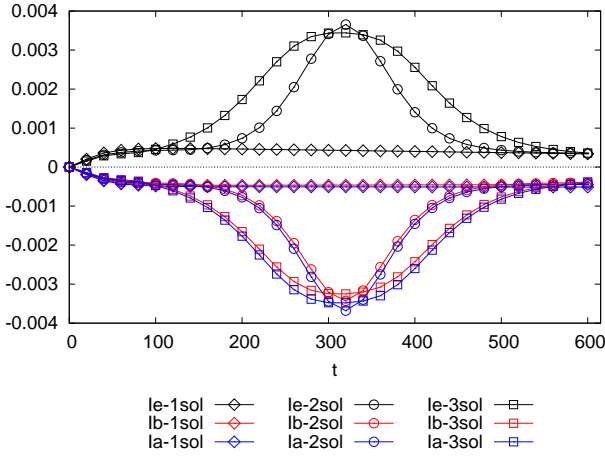


Figure 4: Relative changes of I_a and I_b as a functions of time for the three waves presented in Fig. 2.

should only provide small variations of approximate invariants $I_{ad}^{(2\alpha)}$ and $I_{ad}^{(2\beta)}$.

This expectation is confirmed by Fig. 5. For long time evolution, the relative changes of all approximate invariants are less than of the order of 0.00025.

The precision of numerical calculations of time evolution according to KdV2 can be verified by presentation of the exact invariant, that is volume conservation. Its numerical values displayed in Fig. 6 are constant up to 10 digits.

Similarly as adiabatic invariants $I_{ad}^{(2)}$ presented in Fig. 3, the absolute values of adiabatic invariant $I_{ad}^{(3)}$ are almost constant. It is shown in Fig. 7. Note that the absolute values of $I_{ad}^{(3)}$ are several times smaller than those of $I_{ad}^{(2)}$.

Relative changes of $I_{ad}^{(3)}$ are shown in Fig. 8. They are a little bigger than relative changes of $I_{ad}^{(2)}$.

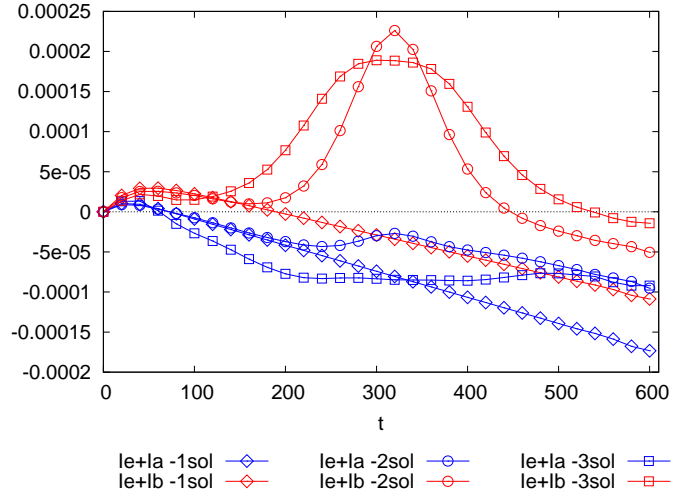


Figure 5: Relative changes of the approximate invariants: $I_{ad}^{(2\alpha)}$, denoted as $I_e + I_a$ and $I_{ad}^{(2\beta)}$ denoted as $I_e + I_b$ for the three waves displayed in the figure 2.

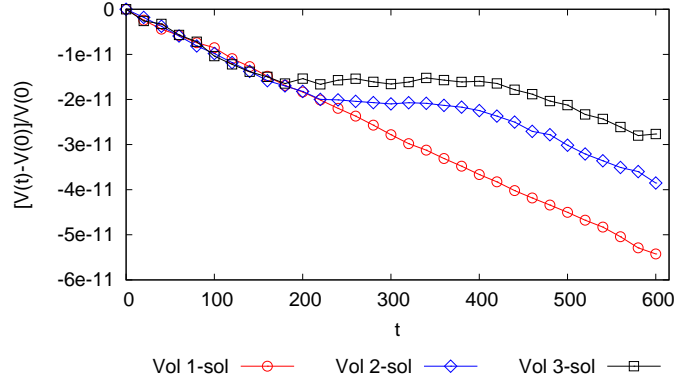


Figure 6: Numerical precision of the volume conservation law for the three waves displayed in Fig. 2.

3.1. Energy and adiabatic invariants

The total energy of the fluid governed by KdV2 equation (1) is given by (see, [10, Eq. (91)])

$$E = E_0 \int_{-\infty}^{\infty} \left[\alpha \eta + (\alpha \eta)^2 + \frac{1}{4} (\alpha \eta)^3 - \frac{3}{32} (\alpha \eta)^4 - \frac{7}{48} \alpha^3 \beta \eta \eta_x^2 \right] dx, \quad (17)$$

where E_0 is given in units of energy.

Relative changes of the energy, that is $(E(t) - E(0))/E(0)$ for time evolution of 1-, 2- and 3-soliton waves, presented in Fig. 2, are displayed in Fig. 9.

How good are adiabatic invariants $I_{ad}^{(2)}$ and $I_{ad}^{(3)}$?

Construct the following combination of the exact invariant $I^{(1)} = \alpha \int_{-\infty}^{\infty} \eta dx$ and adiabatic invariants $I_{ad}^{(2\alpha)}$ and $I_{ad}^{(3)}$

$$\begin{aligned} A_{++} &= \alpha I^{(1)} + \alpha^2 I_{ad}^{(2)} + \alpha^3 I_{ad}^{(3)} \\ &= \int_{-\infty}^{\infty} \left[\alpha \eta + (\alpha \eta)^2 + \frac{1}{4} (\alpha \eta)^3 \right. \\ &\quad \left. + \alpha^3 \left(\eta^3 - \frac{1}{3} \eta_x^2 \right) - (\alpha \eta)^4 - \frac{7}{12} \alpha^4 \eta \eta_x^2 \right] dx \end{aligned} \quad (18)$$

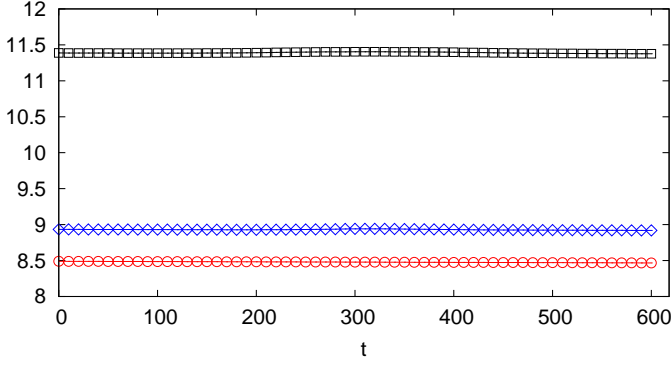


Figure 7: Absolute values of $I_{ad}^{(3)}$ as a function of time for the three waves presented in Fig. 2.

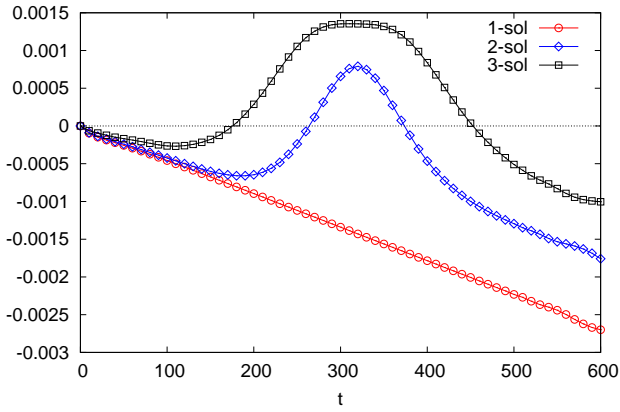


Figure 8: Relative changes of $I_{ad}^{(3)}$ as a function of time for the three waves presented in Fig. 2.

The result is presented in Fig. 10. Here, at least for 2- and 3-soliton waves, the combination of adiabatic invariants (18) is much closer to a constant value than the energy (17).

Apart from volume conservation which holds almost to numerical precision (see, Fig. 6), the adiabatic invariants presented in Figs. 5, 8, 10 and energy shown in Fig. 9 slowly decrease with time. In our opinion the reason lies in fact that initial conditions, taken as 1-, 2-, 3-soliton solutions of KdV equation **are not exact** solutions of KdV2 equation. The single known 1-soliton solution of KdV2 equation found in [15] preserves exactly its shape and then possesses an infinite number of invariants. However, we do not expect the existence of exact n -soliton solutions for KdV2 since it does not belong to a hierarchy of integrable equations. On the other hand the 2- or 3-soliton solutions of an integrable equation like *that obtained through NIT* are likewise not exact solutions of (1) and the deviations from exactness will cause dissipation.

4. Conclusions

- We presented a method of direct calculation of adiabatic invariants for nonintegrable second order equations of KdV type. It can be applied directly for equations written in a fixed reference frame and with several different small parameters of similar order, for instance $\alpha \neq \beta$.

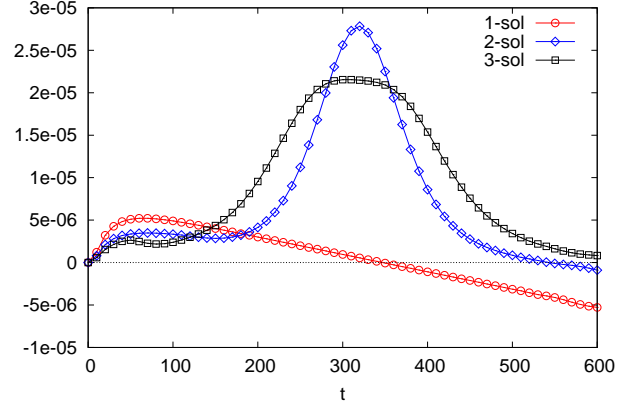


Figure 9: Relative changes of energy as a function of time for the three waves presented in Fig. 2.

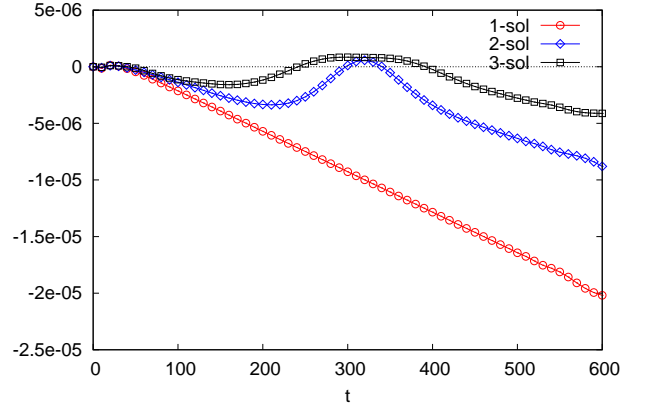


Figure 10: Relative changes of A_{++} for the three waves presented in Fig. 2.

- The method does not require a transformation to a particular moving frame nor a near-identity transformation (see Appendix) and therefore calculations are simpler. It can be applied directly to the original 'physical' equation also in dimensional variables.
- Numerical tests proved that adiabatic invariants related to momentum and energy have indeed almost constant values. The largest deviations from these almost constant values appear during soliton collisions.
- Since in the fixed reference frame the KdV2 equation has nonintegrable form, the energy is not an exact constant (see, e.g. Fig. 9).

Appendix A. Near-identity transformation

All our considerations were performed in a fixed reference frame. They were motivated by two facts. First, as we have pointed out in [10, eq. (39)] even for KdV energy has **noninvariant form** (the same fact was shown, in dimension variables, in the paper of Ali and Kalisch [9]). Second, we aim to study invariants, and asymptotic invariants not only for KdV and KdV2

but also for the KdV2 with uneven bottom, derived in [14, 15]. For this equation only a fixed reference frame makes sense.

When the shallow water problem is considered in a moving frame the situation is different. For KdV, energy is expressed by a linear combination of invariants, see, e.g. [10, eq. (80)]. Higher-order versions of the KdV are not unique as a near-identity transformation can be used to make the higher-order terms take any desired form, as shown in [12, 17, 18, 19, 25] and in many other works.

Consider equation (1), transformed to the reference frame moving with speed 1, setting $\alpha = \beta$ as is usual in KdV theory, and rescaling time by the factor α . Then KdV2 (1) takes the following form

$$\eta_t + \mu\eta\eta_x + \delta\eta_{3x} + \alpha(\delta_1\eta_{5x} + \mu\eta^2\eta_x + \sigma_1\eta\eta_{3x} + \sigma_2\eta_x\eta_{2x}) = 0, \quad (\text{A.1})$$

where $\mu = \frac{3}{2}$, $\delta = \frac{1}{6}$, $\delta_1 = \frac{19}{360}$, $\mu_1 = -\frac{3}{8}$, $\sigma_1 = \frac{23}{24}$ and $\sigma_2 = \frac{5}{12}$. This equations is not unique, as near-identity transformation (NIT) [12, 17, 18, 19, 25]

$$\eta = \eta' + \alpha(a\eta'^2 + b\eta'_{xx}) + \dots \quad (\text{A.2})$$

asymptotically reproduces the same equation for η' , but with altered coefficients

$$\mu' = \mu, \mu'_1 = \mu_1 + a\mu, \delta' = \delta, \delta'_1 = \delta_1, \sigma'_1 = \sigma_1, \sigma'_2 = \sigma_2 + 6a\delta - 2b\mu.$$

In order to have the form $\int \eta'^2 dx$ as an invariant, the equation should be Hamiltonian, implying

$$\sigma'_2 = 2\sigma'_1. \quad (\text{A.3})$$

This is not the case for (A.1), but can be achieved by choice of a, b in the near-identity transformation. That is, choose

$$6a\delta - 2b\mu = 2\sigma'_1 - \sigma'_2, \implies a - 3b = -\frac{1}{8} \quad (\text{A.4})$$

and there is a one-parameter family of such transformations. Since the inverse NIT reads

$$\eta' = \eta - \alpha(a\eta^2 + b\eta_{xx}),$$

it follows that there will then be an asymptotically valid conservation law of the form

$$\begin{aligned} \int_{-\infty}^{\infty} \eta'^2 dx &\simeq \int_{-\infty}^{\infty} [\eta^2 - 2\alpha\eta(a\eta^2 + b\eta_{xx})] dx \\ &= \int_{-\infty}^{\infty} [\eta^2 - 2\alpha a\eta^3 + 2\alpha b\eta_x^2] dx. \end{aligned} \quad (\text{A.5})$$

The forms (9) and (12) of adiabatic invariants obtained in direct calculation can be reproduced by a near identity transformation by choosing either $a = 0$, $b = \frac{1}{24}$ or $a = -\frac{1}{8}$, $b = 0$, respectively. The general one parameter form (13) corresponds to (A.5) where a, b are in linear relation (A.5).

A near-identity transformation also supplies the Hamiltonian for an asymptotically identical second order equations. This Hamiltonian provides the energy invariant of the transformed equation, which is then the adiabatic invariant of the original, nonintegrable equation.

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